

ON THE GAUGE ACTION OF A LEAVITT PATH ALGEBRA

MARÍA GUADALUPE CORRALES GARCÍA, DOLORES MARTÍN BARQUERO,
AND CÁNDIDO MARTÍN GONZÁLEZ

ABSTRACT. We introduce a revised notion of gauge action in relation with Leavitt path algebras. This notion is based on group schemes and captures the full information of the grading on the algebra as it is the case of the gauge action of the graph C^* -algebra of the graph.

1. NOTATIONS

For a graph E denote by $C^*(E)$ the graph C^* -algebra (see for instance [10]) and given any field K , denote by $L_K(E)$ the Leavitt path algebra associated to E (see [2]). As usual for any ring k we will denote by k^\times the group of invertible elements of k . Also denote by $\mathbb{T} := S^1$ the unit circle in \mathbb{R}^2 . When we speak of the canonical \mathbb{Z} -grading on $C^*(E)$ or in $L_K(E)$ we mean the one for which the vertex are of degree 0 and the element $f_1 \cdots f_n g_1^* \cdots g_m^*$ is homogeneous of degree $n - m$ (for any collection of edges f_i and g_j). Now let $A = C^*(E)$ or $L_K(E)$ and consider the canonical \mathbb{Z} -grading $A = \bigoplus_n A_n$. Then for any n we consider the canonical epimorphism $p: \mathbb{Z} \rightarrow \mathbb{Z}_n$ and the grading $A = \bigoplus_{i=0}^{n-1} B_i$ where $B_i = \bigoplus_{p(n)=i} A_n$. This is a coarsening of the canonical \mathbb{Z} -grading and since it is a \mathbb{Z}_n -grading, we call it the canonical \mathbb{Z}_n -grading on A . In path algebras literature, for a path μ in a graph E we denote the source of μ by $s(\mu)$ and the range of μ by $r(\mu)$.

2. DRAWBACKS OF THE CONVENTIONAL DEFINITION

The gauge action of the C^* -algebra $A := C^*(E)$ of a graph E is defined as the group homomorphism $\rho: \mathbb{T} \rightarrow \text{aut}(A)$ such that $\rho(z)(u) = u$ for each vertex u of the graph and $\rho(z)(f) = zf$, $\rho(z)(f^*) = z^{-1}f^*$ for any arrow f and any $z \in \mathbb{T}$ (see [10]). With this definition of the gauge action we can recover the homogeneous components of A easily since for any integer n we have that A_n is just the set of all $a \in A$ such that for any $z \in \mathbb{T}$ we have $\rho(z)(a) = z^n a$. Thus the grading on A induces the gauge action but reciprocally, if we are given the gauge action we reconstruct immediately the homogeneous components of the grading. Since the gauge action of A codifies all the information of the graded algebra

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All the notions related to this graded structure can be defined in terms of the action. That is the reason why the gauge action is omnipresent in the theory of graph C^* -algebras. Most research works on this kind of C^* -algebras involve its gauge action. By contrast, most works on Leavitt path algebras miss the gauge action in the terms in which it has been defined in the literature.

Let us think about the “official” definition of the gauge action of a Leavitt path K -algebra $B := L_K(E)$ (see [1]). This is nothing but the group homomorphism $\tau: K^\times \rightarrow \text{aut}(B)$ such that $\tau(z)(u) = u$, $\tau(z)(f) = zf$ and $\tau(z)(f^*) = z^{-1}f^*$ for any vertex u , any edge f and any $z \in K^\times$.

Drawback 1. *The gauge action τ does not capture the whole information of the graded algebra $L_K(E)$.*

Proof. In fact in some extreme cases τ contains no information at all simply because τ is trivial. For instance take $K = \mathbb{F}_2$ to be the field of two elements. Then K^\times is the trivial group $K^\times = \{1\}$ and τ is the trivial group homomorphism $1 \mapsto 1$. So in this case τ gives no information at all of the grading on $B = L_K(E)$. In other cases in which τ is not trivial, we can not recover the original grading on B . Take for instance $K = \mathbb{F}_3$ so that $K^\times = \{\pm 1\} \cong \mathbb{Z}_2$. Then τ is completely determined by $\tau(-1)$. If we denote by B_n the homogeneous component of degree n of the canonical grading on B , then B_n does not agree with the subspace

$$\{x \in B: \tau(z)(x) = z^n x, \text{ for all } z \in K^\times\}.$$

In fact the above subspace agrees with $\{x \in B: \tau(-1)(x) = (-1)^n x\}$ and is the direct sum of homogeneous components of even degree (if n is even) and the sum of components of odd degree if n is odd. So what we get from the gauge action is the canonical \mathbb{Z}_2 -grading obtained as a coarsening of the canonical \mathbb{Z} -grading of B . By contrast with what happens to the gauge action on $C^*(E)$ we have

$$B_n \subset \{x \in B: \tau(z)(x) = z^n x, \text{ for all } z \in K^\times\}$$

but equality rarely holds (it holds for instance if K is of characteristic 0).

Drawback 2. *For the gauge action ρ of $A = C^*(E)$ the notion of graded ideal is equivalent to that of ρ -invariant ideal. This is not the case for the gauge action τ of $B = L_K(E)$.*

Proof. Recall that an ideal I of A is ρ -invariant if and only if $\rho(z)(I) \subset I$ for all $z \in \mathbb{T}$. Of course when an ideal I of A is a graded ideal then it is ρ -invariant. On the other hand if $\rho(z)(I) \subset I$ for any $z \in \mathbb{T}$ it is easily seen that I is graded by a Vandermonde argument. Indeed if we take $a \in I$ we know that $a = \sum_n a_n$ where $a_n \in A_n$ (homogeneous component of degree n) then $\rho(z)(a) \in I$ hence $\sum_n z^n a_n \in I$ for any $z \in \mathbb{T}$. Thus

$$(1) \quad (a_{-k}, \dots, a_0, a_1, \dots, a_k) \begin{pmatrix} 1 & z_1^{-k} & \cdots & z_{2k}^{-k} \\ 1 & z_1^{-k+1} & \cdots & z_{2k}^{-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_1^k & \cdots & z_{2k}^k \end{pmatrix} \in \overbrace{I \times \cdots \times I}^{2k+1} =: I^{2k+1}$$

for any collection of $z_i \in \mathbb{T}$. Denoting by M the matrix in the right hand member of equation (1) if this matrix were invertible then could conclude that

$$(a_{-k}, \dots, a_0, a_1, \dots, a_k) \in I^{2k+1}$$

hence $a_n \in I$ for every n . But the determinant of M is a Vandermonde determinant which is nonzero if and only if the z_i are all different. For the graph C^* -algebra there is no problem because the ground field is \mathbb{C} . But for the Leavitt path algebra B the possibility of choosing these scalars all different depends on the ground field and it is not always guaranteed. Thus in general we do not have that gauge invariant ideals of B are graded ideals .

Drawback 3. Consider two graph C^* -algebras A_i ($i = 1, 2$) with associated gauge actions ρ_i . Define a homomorphism $f: A_1 \rightarrow A_2$ to be a gauge-homomorphism when for any $z \in \mathbb{T}$ the following square is commutative

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ \rho_1(z) \downarrow & & \downarrow \rho_2(z) \\ A_1 & \xrightarrow{f} & A_2 \end{array}$$

In a similar fashion can we define the notion of a gauge homomorphism of Leavitt path algebras. However while the notion of gauge-homomorphism is equivalent to that of graded homomorphism in the setting of graph C^* -algebras, it is not the case that for Leavitt path algebras, both notions agree.

Proof. In the ambient of Leavitt path algebras the gauge action may be even trivial (if the ground field has characteristic 2). Thus gauge-homomorphisms are simply homomorphism in this case. Since not every homomorphism is graded we see that both notions do not agree. For graph C^* -algebras we can do a Vandermonde argument as before to prove that both notions agree. Of course this drawback and the previous do not exist for Leavitt path algebras over infinite fields but we would like to give a notion of gauge action which overcomes these difficulties and does not depend so much of the ground field.

Some more drawbacks will be explained in the sequel. For the moment we can realize that the canonical \mathbb{Z} -grading of graph C^* -algebras and also in Leavitt path algebras are present in many argumentations on these algebras. Frequently we can argue on homogeneous elements and then generalize to arbitrary ones. That is why is so frequent to find the gauge action in the graph C^* -algebra literature and not in the Leavitt path algebra one (where one must replace gauge action arguments with others involving the canonical \mathbb{Z} -grading).

Drawback 4. The Gauge-Invariant Uniqueness Theorem is stated in [10] in the following terms

Theorem 1. ([10, Theorem 2.2, p.16]) Let E be a row-finite graph and suppose that $\{T, Q\}$ is a Cuntz-Krieger E -family in a C^* -algebra B with each $Q_v \neq 0$. If there is a continuous

action $\beta: \mathbb{T} \rightarrow \text{aut}(B)$ such that $\beta(z)(T_e) = zT_e$ for every $e \in E^1$ and $\beta(z)(Q_v) = Q_v$ for every $v \in E^0$, then $\pi_{T,Q}$ is an isomorphism onto $C^*(T, Q)$.

As far as we know the best version of the previous theorem for Leavitt path algebras is given in [1, Theorem 1.8, p. 6] and it claims:

Theorem 2. (*The Algebraic Gauge-Invariant Uniqueness Theorem.*) Let E be a row-finite graph, let K be an infinite field, and let A be a K -algebra. Suppose $\phi: L_K(E) \rightarrow A$ is a K -algebra homomorphism such that $\phi(v) \neq 0$ for every $v \in E^0$. If there exists a group action $\sigma: K^\times \rightarrow \text{Aut}_K(A)$ such that $\phi \circ \tau_t^E = \sigma_t \circ \phi$ for every $t \in K^\times$, then ϕ is injective.

As we shall see, the hypothesis on the infiniteness of K is not necessary if we use the schematic version of the gauge action.

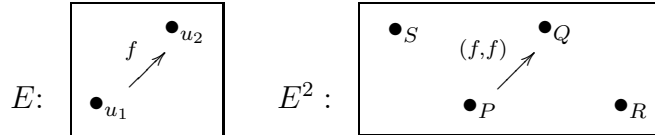
Drawback 5. The natural translation of the crossed product of C^* -algebras to a purely algebraic setting must be made carefully.

We recall the definition of the crossed product of C^* -algebras. Assume that A and B are C^* -algebras and G a compact abelian group with actions $\mu: G \rightarrow \text{aut}(A)$ and $\nu: G \rightarrow \text{aut}(B)$. Consider next the action $\lambda: G \rightarrow \text{aut}(A \otimes B)$ defined by $\lambda(g)(a \otimes b) = \mu(g)(a) \otimes \nu(g^{-1})(b)$. Define now the *crossed product* $A \otimes_G B$ as the fixed point algebra under the action λ .

The gauge action of a graph C^* -algebra has been successfully applied to certain interesting constructions in [3] and [6]. Take two row-finite graphs E and F and define its product $E \times F := (E^0 \times F^0, E^1 \times F^1, r, s)$ where $s(f, g) = (s(f), s(g))$ and $r(f, g) = (r(f), r(g))$ for any $(f, g) \in E^1 \times F^1$. Though this is not the usual definition of the product of two graphs, this notion is interesting for us since it allows to describe the crossed product of graph C^* -algebras. Indeed, it is proved in [3, Proposition 4.1, p. 62] that if E and F are row-finite graphs with no sinks, then there is an isomorphism

$$C^*(E \times F) \cong C^*(E) \otimes_{\mathbb{T}} C^*(F).$$

If we take a naive interpretation of the crossed product of algebras, the similar property for Leavitt path algebras does not hold (for instance if the gauge action of a Leavitt path algebra is trivial then we would have $L_K(E \times E) \cong L_K(E) \otimes_{K^\times} L_K(E)$. But this is not true, since you can get the graph E on the left hand of:



then $E^2 := E \times E$ is the graph on the right hand of the above figure. Thus $L_K(E) \cong \mathcal{M}_2(K)$ and $L_K(E^2) = K \oplus K \oplus \mathcal{M}_2(K)$ which has dimension 6. If the gauge action of $L_K(E)$ is trivial then $L_K(E) \otimes_{K^\times} L_K(E) = L_K(E) \otimes L_K(E)$ which has dimension 16. Hence the cross product of the Leavitt path algebras does not agree with the Leavitt path algebra of E^2 .

Once we have realized some handicaps of the gauge action of a Leavitt path algebras, we propose a different approach.

3. REDEFINING THE GAUGE ACTION

For any associative and commutative ring k (with unit $1 \in k$) denote by \mathbf{alg}_k the category of associative commutative k -algebras and by \mathbf{grp} that of groups. Recall that a k -group functor is a covariant functor $\mathcal{F}: \mathbf{alg}_k \rightarrow \mathbf{grp}$. If \mathcal{F} and \mathcal{G} are k -group functors, a homomorphism $\eta: \mathcal{F} \rightarrow \mathcal{G}$ is nothing but a natural transformation from \mathcal{F} to \mathcal{G} .

Recall also that an affine k -group scheme is a representable k -group functor, that is, $\mathcal{F} = \text{hom}_{\mathbf{alg}_k}(H, -)$ for some Hopf algebra H (see [9] or [5]). An affine group scheme is said to be an algebraic group if the representing Hopf algebra is finitely-generated.

For any k -algebra A we can consider the k -group functor $\mathbf{aut}(A): \mathbf{alg}_k \rightarrow \mathbf{grp}$ such that $\mathbf{aut}(A)(R) := \text{aut}(A_R)$ (where $A_R := A \otimes R$) for any object R in the category \mathbf{alg}_k , that is, for any associative commutative k -algebra R . We also recall the definition of the k -group functor $\mathbf{GL}_n: \mathbf{alg}_k \rightarrow \mathbf{grp}$ such that $\mathbf{GL}_n(R)$ is the group of invertible $n \times n$ matrices with entries in R . In particular $\mathbf{GL}_1(R) = R^\times$ the group of invertible elements in R . This k -group \mathbf{GL}_1 is representable (its representing Hopf algebra being the Laurent polynomial algebra $k[x, x^{-1}]$), hence it is an affine group scheme (and even an algebraic group).

A diagonalizable affine group scheme (diagonalizable group in the sequel) is an affine group scheme whose representing Hopf algebra is the group algebra of an abelian group. Thus if Λ is an abelian group and we consider the group algebra $k\Lambda$ (with its natural structure of Hopf algebra), then the k -group functor $\text{hom}_{\mathbf{alg}_k}(k\Lambda, -)$ is said to be diagonalizable and its usual notation is $\mathbf{Diag}(\Lambda) := \text{hom}_{\mathbf{alg}_k}(k\Lambda, -)$. When Λ is finitely-generated $\mathbf{Diag}(\Lambda)$ is an algebraic group (the group structure in $\text{hom}_{\mathbf{alg}_k}(k\Lambda, R)$ is point-wise multiplication, that is, if $\alpha, \beta \in \text{hom}(k\Lambda, R)$ then $(\alpha\beta)(x) := \alpha(x)\beta(x)$ for any $x \in k\Lambda$).

3.1. Representation of diagonalizable groups. In this section we note a series of results which are well-known but not so easy to quote (at least in its present form). There are two (equivalent) approaches to the study of gradings. Both are based upon affine group schemes. On the one hand we have the co-modules approach which skips the most puzzling notion of affine schemes, and on the second hand we have the representations of diagonalizable groups. The gauge action of C^* -algebras as well as the definition of gauge action for Leavitt path algebras are closer to the viewpoint of representations of diagonalizable group schemes. So we adopt this philosophy.

Most of the materials in this subsection can be seen in [4], [5] and [7] with the slightly different terminology of co-modules. That is why we include this subsection in which we simply translate the main results to the language of representations. Of course the reader familiarized with representations of affine group schemes could skip this subsection and proceed with the next.

Consider a k -module M and define the k -group functor $\mathbf{GL}(M): \mathbf{alg}_k \rightarrow \mathbf{grp}$ such that for any R we have $\mathbf{GL}(M)(R) := \text{GL}(M_R)$ where $M_R := M \otimes R$ and $\text{GL}(M_R)$ is the group of invertible automorphisms of the R -module M_R . If G is a k -group functor and $\rho: G \rightarrow \mathbf{GL}(M)$ is a k -group homomorphism then it is said that ρ is a representation of G . This admits in a standard way a formulation in terms of modules as the reader can guess. For any k -algebra R in \mathbf{alg}_k we have a group homomorphism $\rho_R: G(R) \rightarrow \text{GL}(M_R)$

and if $\alpha: R \rightarrow S$ is a homomorphism of k -algebras there is a commutative diagram

$$\begin{array}{ccc} G(R) & \xrightarrow{\rho_R} & \mathrm{GL}(M_R) \\ G(\alpha) \downarrow & & \downarrow \alpha^* \\ G(S) & \xrightarrow{\rho_S} & \mathrm{GL}(M_S) \end{array}$$

where for any $f \in \mathrm{GL}(M_R)$ the map $\alpha^*(f): M_S \rightarrow M_S$ is given by $\alpha^*(f)(m \otimes 1) = (1 \otimes \alpha)f(m \otimes 1)$. In other words, α^* is the morphism-function of the functor $\mathbf{GL}(M)$.

If G turns out to be a diagonalizable group $G = \mathbf{Diag}(\Lambda)$ and we consider a representation $\rho: G \rightarrow \mathbf{GL}(M)$ then we have group homomorphisms $\rho_R: \mathrm{hom}(k\Lambda, R) \rightarrow \mathrm{GL}(M_R)$ for any k -algebra R . In particular we have $\rho_{k\Lambda}: \mathcal{E}nd(k\Lambda) \rightarrow \mathrm{GL}(M_{k\Lambda})$ and the above commutative diagram specializes to

$$\begin{array}{ccccc} f & & \mathcal{E}nd(k\Lambda) & \xrightarrow{\rho_{k\Lambda}} & \mathrm{GL}(M_{k\Lambda}) \\ \downarrow & & \downarrow & & \downarrow \alpha^* \\ \alpha \circ f & & \mathrm{hom}(k\Lambda, R) & \xrightarrow{\rho_R} & \mathrm{GL}(M_R) \end{array}$$

where $\alpha: k\Lambda \rightarrow R$ is a k -algebra homomorphism. Then the commutativity of the diagram yields (taking $f = 1_{k\Lambda}$) the formula $\rho_R(\alpha) = \alpha^*(\rho_{k\Lambda}(1_{k\Lambda}))$ and so

$$(2) \quad \rho_R(\alpha)(m \otimes 1) = (1 \otimes \alpha)[\rho_{k\Lambda}(1_{k\Lambda})(m \otimes 1)]$$

that is, ρ_R is completely determined by $\rho_{k\Lambda}$ and so the whole action ρ is completely determined once we know $\rho_{k\Lambda}$. If we take $m \in M$ we may write $\rho_{k\Lambda}(1_{k\Lambda})(m \otimes 1) = \sum_{\lambda \in \Lambda} p_\lambda(m) \otimes \lambda$ where $p_\lambda: M \rightarrow M$ is a k -modules homomorphism.

Theorem 3. *The set $\{p_\lambda\}_{\lambda \in \Lambda}$ is a system of orthogonal idempotents of $\mathcal{E}nd_k(M)$ and for any $m \in M$ we have $m = \sum_{\lambda} p_\lambda(m)$.*

Proof. Consider the identity c of the group $\mathcal{E}nd(k\Lambda)$. This is the map such that $c(\lambda) = 1$ for any $\lambda \in \Lambda$. So $\rho_{k\Lambda}(c)$ is the identity on $M_{k\Lambda}$ therefore $m \otimes 1 = \rho_{k\Lambda}(c)(m \otimes 1)$ and applying formula (2) we get

$$\begin{aligned} m \otimes 1 &= \rho_{k\Lambda}(c)(m \otimes 1) = (1 \otimes c)[\rho_{k\Lambda}(1_{k\Lambda})(m \otimes 1)] = \\ &= (1 \otimes c)\left(\sum_{\lambda} p_\lambda(m) \otimes \lambda\right) = \sum_{\lambda} p_\lambda(m) \otimes 1 \end{aligned}$$

whence the second assertion of the Theorem ($m \otimes 1 = 0$ implies $m = 0$ applying 1ϵ where ϵ is the counit of the Hopf algebra $k\Lambda$). To see the first one we use: the equality $\rho_R(\alpha\beta) = \rho_R(\alpha)\rho_R(\beta)$ which holds for any k -algebra R and any $\alpha, \beta \in \mathrm{hom}_{\mathbf{alg}_k}(k\Lambda, R)$. Thus we take $R = k\Lambda \otimes k\Lambda$ and $\alpha, \beta: k\Lambda \rightarrow R$ such that $\alpha(\lambda) = \lambda \otimes 1$, $\beta(\lambda) = 1 \otimes \lambda$ for any $\lambda \in \Lambda$.

Taking into account this as well as equation (2) we get:

$$\sum_{\lambda} p_\lambda(m) \otimes \lambda \otimes \lambda = \sum_{\lambda, \mu} p_\mu p_\lambda(m) \otimes \mu \otimes \lambda$$

and since $\{\mu\phi\lambda: \mu, \lambda \in \Lambda\}$ is a basis of R we conclude that for any $\lambda, \mu \in \Lambda$ one has $p_\lambda^2 = p_\lambda$ and $p_\mu p_\lambda = 0$ if $\lambda \neq \mu$.

Corollary 1. *If the k -module M admits a decomposition $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ where Λ is an abelian group, then there is a representation $\rho: \mathbf{Diag}(\Lambda) \rightarrow \mathbf{GL}(M)$ such that for any k -algebra R the map ρ_R acts in the form $\rho_R(\alpha)(m_\lambda \otimes 1) = m_\lambda \otimes \alpha(\lambda)$ for any $\alpha \in \text{hom}(k\Lambda, R)$ and any $m_\lambda \in M_\lambda$. Reciprocally given a representation $\rho: \mathbf{Diag}(\Lambda) \rightarrow \mathbf{GL}(M)$ of the diagonalizable affine group scheme $\mathbf{Diag}(\Lambda)$, there is a decomposition $M = \bigoplus_\lambda M_\lambda$ where $M_\lambda = p_\lambda(M)$.*

Proof. The only thing to take into account for the reciprocal is that the set of orthogonal idempotents $\{p_\lambda\}$ induce the decomposition $M = \bigoplus_\lambda M_\lambda$.

Now the last result can be adapted to handle group gradings on algebras.

Corollary 2. *If A is a k -graded algebra $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ where Λ is an abelian group, there is a representation $\rho: \mathbf{Diag}(\Lambda) \rightarrow \mathbf{aut}(A)$ such that for any k -algebra R the map ρ_R acts in the form $\rho_R(\alpha)(x_\lambda \otimes 1) = x_\lambda \otimes \alpha(\lambda)$ for any $\alpha \in \text{hom}(k\Lambda, R)$ and any $x_\lambda \in A_\lambda$. Reciprocally given a representation $\rho: \mathbf{Diag}(\Lambda) \rightarrow \mathbf{aut}(A)$ of the diagonalizable affine group scheme $\mathbf{Diag}(\Lambda)$, there is a grading $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ where A_λ is the set of all $x \in A$ such that $\rho_R(\alpha)(x \otimes 1) = x \otimes \alpha(\lambda)$ (for any k -algebra R and $\alpha \in \text{hom}(k\Lambda, R)$).*

3.2. A new concept of gauge action. For a Leavitt path algebra $A = L_K(E)$ over a field K , the gauge action is defined as the map $\rho: K^\times \rightarrow \mathbf{aut}(A)$ such that for any $z \in K^\times$ the automorphism $\rho(z)$ fixes the vertices and $\rho(z)(f) = zf$, $\rho(z)(f^*) = z^{-1}f^*$ for any edge f . The definition we propose implies affine groups schemes. So consider the affine group scheme $\mathbf{GL}_1 = \mathbf{Diag}(\mathbb{Z})$ whose representing Hopf algebra is the group algebra $K\mathbb{Z}$ of \mathbb{Z} which we identify with the Laurent polynomial algebra $K[x, x^{-1}]$. Thus $\mathbf{GL}_1 := \text{hom}(K[x, x^{-1}], -)$ and we have $\mathbf{GL}_1(R) = \text{hom}(K[x, x^{-1}], R) \cong R^\times$ since a K -algebras homomorphism $K[x, x^{-1}] \rightarrow R$ is completely determined by the image of x in R (which is an invertible element in R , and so it belongs to R^\times). Consequently we will identify $\mathbf{GL}_1(R)$ with the group of invertible elements R^\times of R . Obviously $\mathbf{GL}_1(K) = K^\times$.

Definition 1. *For a Leavitt path algebra $A = L_K(E)$ over a field K define the gauge action as a representation of the diagonalizable group scheme \mathbf{GL}_1 given by $\rho: \mathbf{GL}_1 \rightarrow \mathbf{aut}(A)$ where for any K -algebra R and any $z \in R^\times$ we have $\rho_R(z)(u \otimes 1) = u \otimes 1$ for each vertex u and $\rho_R(z)(f \otimes 1) = f \otimes z$, $\rho_R(z)(f^* \otimes 1) = f^* \otimes z^{-1}$ for any $f \in E^1$.*

Whith this schematic approach we see that ρ_K agrees with the “official” definition of gauge action. On the other hand we can define such an action for any K -algebra A endowed with a \mathbb{Z} -grading $A = \bigoplus_{n \in \mathbb{Z}} A_n$: just define for any K -algebra R the map

$$\rho_R(z)(a_n \otimes 1) := a_n \otimes z^n$$

for any $n \in \mathbb{Z}$, $a_n \in A_n$ and any $z \in R^\times$. Reciprocally if we have a representation $\rho: \mathbf{GL}_1 \rightarrow \mathbf{aut}(A)$ for some K -algebra A , then applying Corollary 2, A is \mathbb{Z} -graded where for each integer n we have

$$A_n = \{a \in A: \rho_R(z)(a \otimes 1) = a \otimes z^n \text{ for all } z \in R^\times \text{ and each object } R \text{ in } \mathbf{alg}_K\}.$$

The first conclusion we get is

Theorem 4. *The gauge action in schematic sense encloses all the information of the grading. We can recover the homogeneous components from the schematic gauge action. So Drawback 1 no longer holds with this new definition.*

Let us go now to the notion of ρ -invariant ideal of $A := L_K(E)$. So we assume given $\rho: \mathbf{GL}_1 \rightarrow \mathbf{aut}(A)$ the gauge action in schematic sense.

Definition 2. *An ideal I of A is said to be ρ -invariant when for any K -algebra R and any $z \in R^\times$ we have $\rho_R(z)(I \otimes 1) \subset I \otimes R$.*

Clearly, if I is a graded ideal of A then I is ρ -invariant: indeed take a K -algebra R and any $z \in R^\times$. Take any $a \in I$, then $a = \sum a_n$ where each $a_n \in I \cap A_n$. Thus $\rho_R(z)(a \otimes 1) = \sum_n \rho_R(z)(a_n \otimes 1) = \sum_n a_n \otimes z^n \in I \otimes R$. Consequently graded ideals of A are ρ -invariant. But the reciprocal is also true:

Theorem 5. *An ideal I of A is graded if and only if it is ρ -invariant. Thus Drawback 2 no longer holds.*

Proof. If $\rho_R(z)(I \otimes 1) \subset I \otimes R$ for any R and $z \in R^\times$, take $a \in I$ and its decomposition $a = \sum_{-k}^k a_n$ where $a_n \in A_n$. Then $\rho_R(z)(a \otimes 1) = \sum_n a_n \otimes z^n \in I \otimes R$ for any $z \in R^\times$. Now take R to be the algebraic closure of K . Then choosing a collection $z_{-k}, \dots, z_k \in R^\times$ of pairwise different elements we have:

$$(3) \quad (a_{-k} \otimes 1, \dots, a_k \otimes 1) \begin{pmatrix} 1 & 1\emptyset z_1^{-k} & \cdots & 1\emptyset z_{k-1}^{-k} \\ \vdots & \vdots & & \vdots \\ 1 & 1\emptyset z_1^k & \cdots & 1\emptyset z_{k-1}^k \end{pmatrix} \in \overbrace{(I \otimes R) \times \cdots \times (I \otimes R)}^{2k+1} =: (I \otimes R)^{2k+1}$$

where the matrix is invertible (again Vandermonde's argument). If we denote by M such matrix then we have $(a_{-k} \otimes 1, \dots, a_k \otimes 1) \in M^{-1}(I \otimes R)^{2k+1} \subset (I \otimes R)^{2k+1}$ which proves that for any n we have $a_n \otimes 1 \in I \otimes R$. This implies that $a_n \in I$ for each n .

Let us deal with Drawback # 3 now. Given two Leavitt path K -algebras A_1 and A_2 with their respective gauge actions in schematic sense $\rho_i: \mathbf{GL}_1 \rightarrow \mathbf{aut}(A_i)$, $i = 1, 2$. Then

Definition 3. *A homomorphism $f: A_1 \rightarrow A_2$ is said to be a gauge-homomorphism if the following square is commutative.*

$$\begin{array}{ccc} A_1 \otimes R & \xrightarrow{f \otimes 1} & A_2 \otimes R \\ (\rho_1)_R(z) \downarrow & & \downarrow (\rho_2)_R(z) \\ A_1 \otimes R & \xrightarrow{f \otimes 1} & A_2 \otimes R \end{array}$$

for any K -algebra R and any $z \in R^\times$.

It is easy to prove that in case $f: A_1 \rightarrow A_2$ is a graded homomorphism, then it is a gauge-homomorphism: take $a \in A_1$ homogeneous of degree say n . Then $(\rho_2)_R(z)(f \otimes 1)(a \otimes 1) = (\rho_2)_R(z)(f(a) \otimes 1)$ and since $f(a)$ is an homogeneous element of A_2 of degree n then $(\rho_2)_R(z)(f \otimes 1)(a \otimes 1) =$

$f(a)\emptyset z^n = (f\emptyset 1)(a\emptyset z^n) = (f\emptyset 1)(\rho_1)_R(z)(a\emptyset 1)$. Thus f is a gauge-homomorphism. But we have also the reciprocal.

Theorem 6. *The homomorphism $f: A_1 \rightarrow A_2$ is graded if and only if it is a gauge-homomorphism. Thus Drawback 3 disappears.*

Proof. The proof is based on the possibility of taking R in \mathbf{alg}_K to be the algebraic closure of K which is an infinite field and then an argument in same line as above (proof of Theorem 5).

The following problem we found in the standard definition of gauge action was the statement of the Algebraic Gauge-Invariant Uniqueness Theorem for Leavitt path algebras. We can re-state it in the following form.

Theorem 7. *(The Schematic Algebraic Gauge-Invariant Uniqueness Theorem.) Let E be a row-finite graph, let K be any field, and let A be a K -algebra. Denote by*

$$\rho: \mathbf{GL}_1 \rightarrow \mathbf{aut}(L_K(E))$$

the gauge-action of $L_K(E)$. Suppose $\phi: L_K(E) \rightarrow A$ is a K -algebra homomorphism such that $\phi(v) \neq 0$ for every $v \in E^0$. If there exists an action $\sigma: \mathbf{GL}_1 \rightarrow \mathbf{aut}(A)$ such that $(\phi\emptyset 1)\rho_R(z) = \sigma_R(z)(\phi\emptyset 1)$ for every $z \in R^\times$, then ϕ is injective.

In this theorem we eliminate the hypothesis on the infiniteness of K (however the schematic version of the gauge action must be used instead of the standard one). The proof is straightforward since by Theorem 6 the homomorphism ϕ is graded and then we can apply [8, Theorem 4.8].

Also by using the gauge action in schematic sense the hypothesis on the infiniteness of the ground field K in [1, Proposition 1.6] can be dropped. Since the notion of graded ideal and of gauge invariant ideal agree when we use the schematic version of the gauge action, such exceptionalities as the ones observed in [1, Proposition 1.7] are no longer present.

3.3. Cross product of algebras. Let K be a field and A a K -algebra with an action $\rho: G \rightarrow \mathbf{aut}(A)$ where G is an affine group scheme. This means that ρ is a natural transformation between the given K -group functors. Then we define

Definition 4. *The fixed subalgebra A^ρ of A under ρ is the one whose elements are the elements $a \in A$ such that $\rho_R(z)(a \otimes 1) = a\emptyset 1$ for any K -algebra R and any $z \in G(R)$.*

If A and B are K -algebras provided with actions $\rho: G \rightarrow \mathbf{aut}(A)$ and $\sigma: G \rightarrow \mathbf{aut}(B)$ then there is an action $\rho\emptyset\sigma: G \rightarrow \mathbf{aut}(A\emptyset B)$ such that for any K -algebra R and any $z \in G(R)$ we have $(\rho\emptyset\sigma)_R(z)$ given by the composition

$$\begin{array}{ccccc} (A\emptyset B)_R & = & A\emptyset B\emptyset R & \xrightarrow{1\emptyset\delta} & A\emptyset B\emptyset R\emptyset R & \xrightarrow{\theta} & A_R\emptyset B_R \\ \downarrow (\rho\emptyset\sigma)_R(z) & & & & & & \downarrow \rho_R(z)\emptyset\sigma_R(z^{-1}) \\ (A\emptyset B)_R & = & A\emptyset B\emptyset R & \xleftarrow{1\emptyset\mu} & A\emptyset B\emptyset R\emptyset R & \xleftarrow{\theta^{-1}} & A_R\emptyset B_R \end{array}$$

where:

- $\delta: R \rightarrow R\emptyset R$ is given by $\delta(z) = z\emptyset 1$,
- θ is the isomorphism $a\emptyset b\emptyset r\emptyset r' \mapsto a\emptyset r\emptyset b\emptyset r'$,
- $\mu: R\emptyset R \rightarrow R$ is the multiplication $\mu(r\emptyset r') = rr'$.

Summarizing

$$(\rho\emptyset\sigma)_R(z) = (1\emptyset\mu)\theta^{-1}(\rho_R(z)\emptyset\sigma_R(z^{-1}))\theta(1\emptyset\delta).$$

Now a direct (but not short) computation reveals that

$$(\rho\emptyset\sigma)_R(zz') = (\rho\emptyset\sigma)_R(z)(\rho\emptyset\sigma)_R(z')$$

for any z and z' . So any $(\rho\emptyset\sigma)_R(z)$ is invertible with inverse $(\rho\emptyset\sigma)_R(z^{-1})$. Moreover, since $(\rho\emptyset\sigma)_R(z)$ is a composition of R -algebras homomorphisms, then $(\rho\emptyset\sigma)_R(z) \in \text{aut}((A\emptyset B)_R)$.

Definition 5. *The action $\rho\emptyset\sigma: G \rightarrow \mathbf{aut}(A \otimes B)$ will be called the tensor product action of ρ and σ . The fixed point subalgebra $(A\emptyset B)^{\rho\emptyset\sigma}$ of $A\emptyset B$ under $\rho\emptyset\sigma$ will be denoted $A \emptyset_\sigma B$ and called the cross product of A and B by the actions ρ and σ . If there is no ambiguity with respect to the actions involved we could shorten the notation to $A\emptyset_G B$.*

It can be proved that this definition is coherent with the previously mentioned for C^* -algebras.

Consider now two Leavitt path algebras $L_K(E)$ and $L_K(F)$ of the graphs E and F respectively. We assume given the gauge action of each algebra and ask about the cross product algebra $L_K(E)\emptyset_{\mathbf{GL}_1} L_K(F)$. With not much effort one can prove that it consists on the elements of the form $\sum_{n \in \mathbb{Z}} a_n \emptyset b_n$ where $a_n \in L_K(E)$ with $\deg(a) = n$ while $b_n \in L_K(F)$ has also degree n . Of course we can define on this algebra also an action $\tau: \mathbf{GL}_1 \rightarrow \text{aut}(L_K(E)\emptyset_{\mathbf{GL}_1} L_K(F))$ by declaring for any k -algebra R and each $z \in R^\times$ that $\tau_R(z)(a_n \otimes r\emptyset b_n \emptyset s) = a_n \emptyset z^n r\emptyset b_n \emptyset s$. This action induces a grading on $L_K(E)\emptyset_{\mathbf{GL}_1} L_K(F)$ in which the homogeneous component of degree n is the space generated by the elements of the form $a\emptyset b$ where a and b are homogeneous of degree n .

Our next goal is to prove

Theorem 8. *If E and F are row-finite graphs with no sinks, then there is an isomorphism $L_K(E)\emptyset_{\mathbf{GL}_1} L_K(F) \cong L_K(E \times F)$ where the product of the graph is the one described in Section 2.*

Proof. We consider the path algebra $K(E \times F)$ whose basis is the set of all paths of the graph $E \times F$. There is a canonical homomorphism of K -algebras map

$$K(E \times F) \rightarrow L_K(E)\emptyset_{\mathbf{GL}_1} L_K(F)$$

such that for any $(u, v) \in E^0 \times F^0$ and $(f, g) \in E^1 \times F^1$ we have

$$(u, v) \mapsto u\emptyset v, \quad (f, g) \mapsto f\emptyset g, \quad (f^*, g^*) \mapsto f^*\emptyset g^*.$$

This homomorphism induces one $\phi: L_K(E \times F) \rightarrow L_K(E)\emptyset_{\mathbf{GL}_1} L_K(F)$ such that $\phi(u, v) \neq 0$ for each $(u, v) \in E^0 \times F^0$. Furthermore if we take the action

$$\tau: \mathbf{GL}_1 \rightarrow \text{aut}(L_K(E)\emptyset_{\mathbf{GL}_1} L_K(F))$$

defined above, we see that $(\phi \otimes 1)\rho_R(z) = \tau_R(z)(\phi\emptyset 1)$ where ρ is the Gauge action of $L_K(E \times F)$. Thus applying Theorem 7 we conclude that ϕ is a monomorphism. To see

that it is also an epimorphism we need the hypothesis that the graphs have no sinks. Since $L_K(E)\phi_{\mathbf{GL}_1}L_K(F)$ is generated by elements of the form $a\phi b$ where $\deg(a) = \deg(b)$ it suffices to show that these elements are in the image of ϕ . First we prove that if μ and τ are paths of the same length (say n) and u is a vertex, then $\mu\tau^*\phi u$ is in the image of ϕ : Indeed, $\mu\tau^*\phi u = \mu\tau^*\phi \sum_i g_i g_i^*$ (since F is row-finite and has no sink). If $\mu = f\mu'$ where $f \in E^1$ and μ' is a path then $\mu\tau^*\phi u = \sum_i (f\phi g_i)(\mu'\tau^*\phi g_i^*)$ and if $\tau = h\tau'$ with $h \in E^1$ and τ' a path then $\mu\tau^*\phi u = \sum_i (f\phi g_i)(\mu'\tau'^*h^*\phi g_i^*) = \sum_i (f\phi g_i)(\mu'\tau'^*\phi r(g_i))(h^*\phi g_i^*)$. Applying a suitable induction hypothesis this proves that $\mu\tau^*\phi u$ is in the image of ϕ . Symetrically it can be proved that the image of ϕ contains the elements of the form $v\phi\sigma\delta^*$ with $v \in E^0$ and σ, δ being paths of F of the same degree. Now any generator of $L_K(E)\phi_{\mathbf{GL}_1}L_K(F)$ say $\mu\tau^*\phi\sigma\delta^*$ such that $\deg(\mu) - \deg(\tau) = \deg(\sigma) - \deg(\delta)$ we can write as a product of elements which obey some of the following patterns:

- $f\phi g$ with $f \in E^1$ and $g \in F^1$.
- $f^*\phi g^*$ with $f \in E^1$ and $g \in F^1$.
- $\mu\tau^*\phi u$ with $u \in F^0$, μ and τ being paths of E of the same length.
- $v\phi\sigma\delta^*$ with $v \in E^0$, σ and δ being paths of F of the same length.

Since any of these elements is in the image of ϕ , this proves that ϕ is an epimorphism.

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M. G. CORRALES: UNIVERSIDAD DE MÁLAGA, DEPARTAMENTO DE ÁLGEBRA GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, CAMPUS DE TEATINOS S/N, 29071 MÁLAGA, SPAIN.

E-mail address: 0618459402@uma.es

D. MARTÍN: DEPARTAMENTO DE MATEMÁTICA APLICADA, ESCUELA TÉCNICA SUPERIOR DE INGENIEROS INDUSTRIALES, UNIVERSIDAD DE MÁLAGA, 29071 MÁLAGA, SPAIN.

E-mail address: dmartin@uma.es

C. MARTÍN: UNIVERSIDAD DE MÁLAGA, DEPARTAMENTO DE ÁLGEBRA GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, CAMPUS DE TEATINOS S/N, 29071 MÁLAGA, SPAIN.

E-mail address: candido@apncs.cie.uma.es